

Discussion 10

1. Let $\tau > 0$ and let X_1, X_2, \dots, X_n be a random sample from the distribution with probability density function

$$f(x; \tau) = \frac{\tau^5}{8} x^{14} e^{-\tau x^3}, \quad x > 0.$$

Obtain the maximum likelihood estimator of τ , $\hat{\tau}$.

2. Let X_1, X_2, \dots, X_n be a random sample from the distribution with probability density function

$$f(x) = \frac{2(\theta - x)}{\theta^2} \quad 0 < x < \theta \quad \theta > 0.$$

- a) Obtain the method of moments estimator of θ , $\tilde{\theta}$.
- b) Is $\tilde{\theta}$ an unbiased estimator for θ ? c) Find $\text{Var}(\tilde{\theta})$.

3. Let X_1, X_2, \dots, X_n be a random sample of size n from the distribution with probability density function

$$f(x; \lambda) = \frac{2\sqrt{\lambda}}{\sqrt{\pi}} e^{-\lambda x^2}, \quad x > 0, \quad \lambda > 0.$$

- a) Obtain the maximum likelihood estimator of λ , $\hat{\lambda}$.
- b) Suppose $n = 4$, and $x_1 = 0.2$, $x_2 = 0.6$, $x_3 = 1.1$, $x_4 = 1.7$.
Find the maximum likelihood estimate of λ .
- c) Obtain the method of moments estimator of λ , $\tilde{\lambda}$.
- d) Suppose $n = 4$, and $x_1 = 0.2$, $x_2 = 0.6$, $x_3 = 1.1$, $x_4 = 1.7$.
Find a method of moments estimate of λ .
- e) Find a closed-form expression for $E(X^k)$, $k > -1$.

4. Let $\lambda > 0$ and let X_1, X_2, \dots, X_n be a random sample from the distribution with probability density function

$$f(x; \lambda) = \frac{\lambda^2}{2} e^{-\lambda\sqrt{x}}, \quad x > 0, \quad \text{zero otherwise.}$$

- a) Find the maximum likelihood estimator of λ , $\hat{\lambda}$.
- b) Suppose $n = 4$, and $x_1 = 0.81$, $x_2 = 1.96$, $x_3 = 0.36$, $x_4 = 0.09$. Find the maximum likelihood estimate of λ , $\hat{\lambda}$.
- c) Find a closed-form expression for $E(X^k)$ for $k > -1$.

“Hint” 1: $u = \sqrt{x}$.

“Hint” 2: $\frac{\lambda^\alpha}{\Gamma(\alpha)} u^{\alpha-1} e^{-\lambda u}$ is the p.d.f. of Gamma(α , $\theta = \frac{1}{\lambda}$) distribution

- d) Find $E(X)$ and $\text{Var}(X)$.
- e) Find a method of moments estimator of λ , $\tilde{\lambda}$.
- f) Suppose $n = 4$, and $x_1 = 0.81$, $x_2 = 1.96$, $x_3 = 0.36$, $x_4 = 0.09$. Find a method of moments estimate of λ , $\tilde{\lambda}$.

5. Let $\lambda > 0$ and let X_1, X_2, \dots, X_n be independent random variables, each with the probability density function

$$f(x; \lambda) = \frac{\lambda}{x^{\lambda+1}}, \quad x > 1.$$

- a) (i) Find the maximum likelihood estimator of λ , $\hat{\lambda}$.
- (ii) Suppose $n = 5$, and $x_1 = 1.3$, $x_2 = 1.4$, $x_3 = 2$, $x_4 = 3$, $x_5 = 5$. Find the maximum likelihood estimate of λ .

- b) (i) Find a method of moments estimator of λ , $\tilde{\lambda}$. (Assume $\lambda > 1$.)
- (ii) Suppose $n = 5$, and $x_1 = 1.3$, $x_2 = 1.4$, $x_3 = 2$, $x_4 = 3$, $x_5 = 5$.
Find a method of moments estimate of λ .

6. Let X_1, X_2, \dots, X_n be a random sample from the distribution with probability mass function

$$P(X_i = 1) = \frac{\theta}{3 + \theta}, \quad P(X_i = 2) = \frac{2}{3 + \theta}, \quad P(X_i = 3) = \frac{1}{3 + \theta}, \quad \theta > 0.$$

- a) Obtain the method of moments estimator $\tilde{\theta}$ of θ .
- b) Obtain the maximum likelihood estimator $\hat{\theta}$ of θ .

7. Bert and Ernie find a coin on the sidewalk on Sesame Street. They wish to estimate p , the probability of Heads. Bert got X Heads in N coin tosses (N is fixed, X is random). Ernie got Heads for the first time on the Y^{th} coin toss (Y is random). They decide to combine their information in hope of a better estimate. (Assume independence.)

- a) What is the likelihood function $L(p) = L(p; X, N, Y)$?
- b) Obtain the maximum likelihood estimator for p .
- c) Explain intuitively why your estimator makes good sense.

8. Let $\theta \in \mathbb{R}$ and let X_1, X_2, \dots, X_n be a random sample from the distribution with probability density function

$$f(x; \theta) = \frac{1}{2} e^{-|x - \theta|}, \quad x \in \mathbb{R}.$$

- a) Find a method of moments estimator $\tilde{\theta}$ of θ .
- b) Find the maximum likelihood estimator $\hat{\theta}$ of θ .