STAT 400
UIUC

1.1 – Properties of Probability

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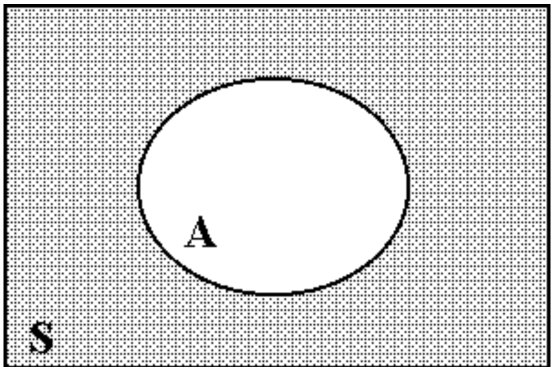
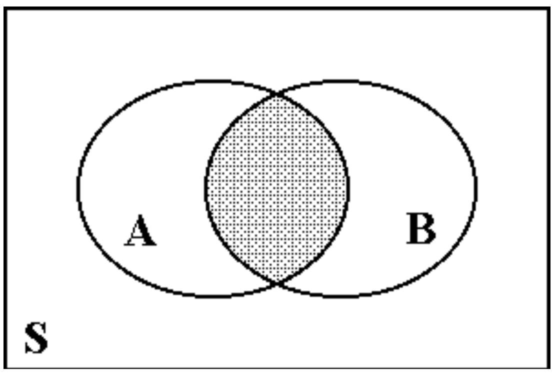


Complement of A

A'

(not A, $\overline{A}$, A^c)

contains all elements that are **<u>not</u>** in A

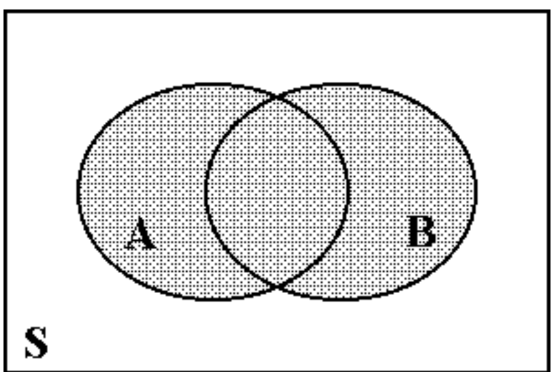


Intersection of A and B

$A \cap B$

(A and B)

contains all elements that are in A **<u>and</u>** in B



Union of A and B

$A \cup B$

(A or B)

contains all elements that are either in A **<u>or</u>** in B or both

<u>Example 1</u>:
Suppose a 6-sided die is rolled. The sample space, S, is { 1, 2, 3, 4, 5, 6 }.
Consider the following events:

A = { the outcome is even },

B = { the outcome is greater than 3 },

a) List outcomes in A, B, A', $A \cap B$, $A \cup B$.

Axiom 1 Let A be any event defined over S. Then $P(A) \geq 0$.

Axiom 2 $P(S) = 1$.

Axiom 3 If $A_1, A_2, A_3, \ldots$ are events and $A_i \cap A_j = \varnothing$ for each $i \neq j$, then

$$P(A_1 \cup A_2 \cup \ldots \cup A_k) = P(A_1) + P(A_2) + \ldots + P(A_k)$$

for each positive integer k, and

$$P(A_1 \cup A_2 \cup A_3 \cup \ldots) = P(A_1) + P(A_2) + P(A_3) + \ldots$$

for an infinite, but countable, number of events.

Theorem 1. $P(A') = 1 - P(A)$.

Theorem 2. $P(\varnothing) = 0$.

Theorem 3. If $A \subseteq B$, then $P(A) \leq P(B)$.

Theorem 4. For any event A, $P(A) \leq 1$.

!	For any event A, $0 \leq P(A) \leq 1$	!	$P(S) = 1$, where S is the sample space.	!

<u>Example 1</u>: (cont.)

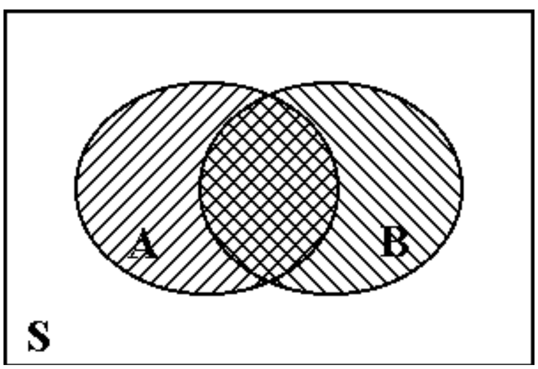
b) Find the probabilities $P(A)$, $P(B)$, $P(A')$, $P(A \cap B)$, $P(A \cup B)$ if the die is fair.

c) Suppose the die is loaded so that the probability of an outcome is proportional to the outcome, i.e.

$$P(1) = p,\quad P(2) = 2p,\quad P(3) = 3p,\quad P(4) = 4p,\quad P(5) = 5p,\quad P(6) = 6p.$$

i) Find the value of p that would make this a valid probability model.

ii) Find the probabilities $P(A)$, $P(B)$, $P(A')$, $P(A \cap B)$, $P(A \cup B)$.



Theorem 5.

If A and B are any two events, then

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

<u>Example 1</u>: (cont.)

Theorem 6.

$$
\begin{aligned}
P(A \cup B \cup C) = {} & P(A) + P(B) + P(C) \\
& - P(A \cap B) - P(A \cap C) - P(B \cap C) \\
& + P(A \cap B \cap C)
\end{aligned}
$$

$$
\begin{aligned}
P(A \cup B \cup C \cup D) = {} & P(A) + P(B) + P(C) + P(D) \\
& - P(A \cap B) - P(A \cap C) - P(A \cap D) \\
& - P(B \cap C) - P(B \cap D) - P(C \cap D) \\
& + P(A \cap B \cap C) + P(A \cap B \cap D) \\
& + P(A \cap C \cap D) + P(B \cap C \cap D) \\
& - P(A \cap B \cap C \cap D)
\end{aligned}
$$

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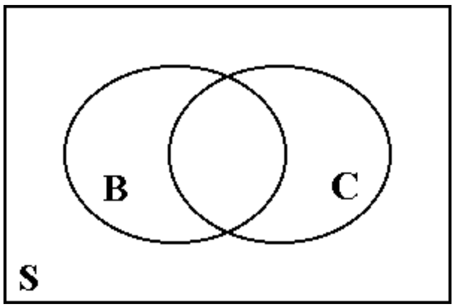
Example 2:
The probability that a randomly selected student at Anytown College owns a bicycle is 0.55, the probability that a student owns a car is 0.30, and the probability that a student owns both is 0.10.



a) What is the probability that a student selected at random does not own a bicycle?

b) What is the probability that a randomly selected student owns either a car or a bicycle, or both?

Law of Total Probability. $P(A) = P(A \cap B) + P(A \cap B')$

De Morgan's Law. $(A \cup B)' = A' \cap B'$
$(A \cap B)' = A' \cup B'$

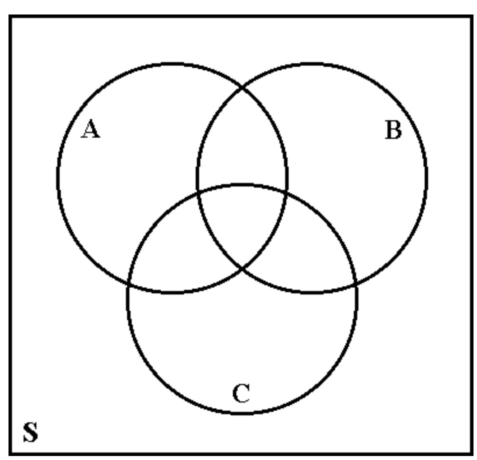
Example 2: (cont.)
c) What is the probability that a student selected at random has neither a car nor a bicycle?

	C	C ′	
B			
B′			

Distributive Law. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Example 3:
Suppose

$P(A) = 0.22$,
$P(B) = 0.25$,
$P(C) = 0.28$,
$P(A \cap B) = 0.11$,
$P(A \cap C) = 0.05$,
$P(B \cap C) = 0.07$,
$P(A \cap B \cap C) = 0.01$.



Find the following:

a) $P(A \cup B)$

b) $P(A' \cap B')$

c) $P(A \cup B \cup C)$

d) $P(A' \cap B' \cap C')$

e) $P(A' \cap B' \cap C)$

f) $P((A' \cap B') \cup C)$

g) $P((A \cup B) \cap C)$

h) $P((B \cap C') \cup A')$

Example 4: During the first week of the semester, 80% of customers at a local convenience store bought either beer or potato chips (or both). 60% bought potato chips. 30% of the customers bought both beer and potato chips. What proportion of customers bought beer?

<u>Example 5</u>:

Let $a > 2$. Suppose S = { 0, 1, 2, 3, … } and

$$P(0) = c, \qquad P(k) = \frac{1}{a^k}, \quad k = 1, 2, 3, \ldots .$$

a) Find the value of c (c will depend on a) that makes this is a valid probability distribution.

Hint: Infinite series reading

b) Find the probability of an odd outcome.

<u>Example 6</u>:

Suppose S = { 0, 1, 2, 3, … } and

$$P(0) = p, \qquad P(k) = \frac{1}{2^k \cdot k!}, \quad k = 1, 2, 3, \ldots .$$

Find the value of p that would make this a valid probability model.

Hint: Infinite series reading