STAT 400
UIUC

2.3 – Moment Generating Function

Stepanov
Dalpiaz
Nguyen

The k^{th} moment of X (the k^{th} moment of X about the origin), μ_k, is given by

$$\mu_k = \mathrm{E}(\mathrm{X}^k) = \sum_{\text{all } x} x^k \cdot f(x)$$

The k^{th} central moment of X (the k^{th} moment of X about the mean), μ_k', is given by

$$\mu_k' = \mathrm{E}((\mathrm{X} - \mu)^k) = \sum_{\text{all } x} (x - \mu)^k \cdot f(x)$$

The moment-generating function of X, $\mathrm{M}_{\mathrm{X}}(t)$, is given by

$$\mathrm{M}_{\mathrm{X}}(t) = \mathrm{E}(e^{t\mathrm{X}}) = \sum_{\text{all } x} e^{tx} \cdot f(x)$$

Theorem 1: $\mathrm{M}_{\mathrm{X}}'(0) = \mathrm{E}(\mathrm{X})$ $\quad$ $\mathrm{M}_{\mathrm{X}}''(0) = \mathrm{E}(\mathrm{X}^2)$

$$\mathrm{M}_{\mathrm{X}}^{(k)}(0) = \mathrm{E}(\mathrm{X}^k)$$

Theorem 2: $\mathrm{M}_{\mathrm{X}_1}(t) = \mathrm{M}_{\mathrm{X}_2}(t)$ for some interval containing 0

$$\Rightarrow \quad f_{\mathrm{X}_1}(x) = f_{\mathrm{X}_2}(x)$$

Theorem 3: Let $\mathrm{Y} = a\mathrm{X} + b$. Then $\mathrm{M}_{\mathrm{Y}}(t) = e^{bt}\mathrm{M}_{\mathrm{X}}(at)$

Example 1:
Suppose a random variable X has the following probability distribution:

x	$f(x)$
10	0.20
11	0.40
12	0.30
13	0.10

Find the moment-generating function of X, $\mathrm{M}_{\mathrm{X}}(t)$.

<u>Example 2</u>:

Suppose the moment-generating function of a random variable X is

$$M_X(t) = 0.10 + 0.15\,e^{t} + 0.20\,e^{2t} + 0.25\,e^{-3t} + 0.30\,e^{5t}.$$

Find the expected value of X, E(X).

<u>Example 3</u>:

Suppose a discrete random variable X has the following probability distribution:

$$f(0) = P(X = 0) = 2 - e^{1/2}, \qquad f(k) = P(X = k) = \frac{1}{2^{k} \cdot k!}, \quad k = 1, 2, 3, \ldots$$

a) Find the moment-generating function of X, $M_X(t)$.

b) Find the expected value of X, $E(X)$, and the variance of X, $\text{Var}(X)$.

<u>Example 4</u>:

Let X be a $\text{Binomial}(n, p)$ random variable.

Find the moment-generating function of X.

<u>Example 5</u>:

Let X be a geometric random variable with probability of "success" p.

a) Find the moment-generating function of X.

b) Use the moment-generating function of X to find $E(X)$.

Example 6:

a) Find the moment-generating function of a Poisson random variable.

Consider $\ln M_X(t)$. (cumulant generating function)

$$(\ln M_X(t))' = \frac{M_X'(t)}{M_X(t)} \qquad (\ln M_X(t))'' = \frac{M_X''(t) \cdot M_X(t) - \left[M_X'(t)\right]^2}{[M_X(t)]^2}$$

Since $M_X(0) = 1$, $M_X'(0) = E(X)$, $M_X''(0) = E(X^2)$,

$$(\ln M_X(t))' \big|_{t=0} = E(X) = \mu_X$$

$$(\ln M_X(t))'' \big|_{t=0} = E(X^2) - [E(X)]^2 = \sigma_X^2$$

b) Find $E(X)$ and $Var(X)$, where X is a Poisson random variable.