

Covariance of X and Y

$$\sigma_{XY} = \text{Cov}(X, Y) = E[(X - \mu_X)(Y - \mu_Y)] = E(XY) - \mu_X \mu_Y$$

- (a)  $\text{Cov}(X, X) = \text{Var}(X)$ ;
- (b)  $\text{Cov}(X, Y) = \text{Cov}(Y, X)$ ;
- (c)  $\text{Cov}(aX + b, Y) = a \text{Cov}(X, Y)$ ;
- (d)  $\text{Cov}(X + Y, W) = \text{Cov}(X, W) + \text{Cov}(Y, W)$ .

$$\begin{aligned} \text{Cov}(aX + bY, cX + dY) \\ = ac \text{Var}(X) + (ad + bc) \text{Cov}(X, Y) + bd \text{Var}(Y). \end{aligned}$$

$$\begin{aligned} \text{Var}(aX + bY) = \text{Cov}(aX + bY, aX + bY) \\ = a^2 \text{Var}(X) + 2ab \text{Cov}(X, Y) + b^2 \text{Var}(Y). \end{aligned}$$

**Example 0:**

Find in terms of  $\sigma_X^2$ ,  $\sigma_Y^2$ , and  $\sigma_{XY}$ :

a)  $\text{Cov}(2X + 3Y, X - 2Y)$

b)  $\text{Var}(2X + 3Y)$

c)  $\text{Var}(X - 2Y)$

Correlation coefficient of X and Y

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)} \cdot \sqrt{\text{Var}(Y)}} = E \left[ \left( \frac{X - \mu_X}{\sigma_X} \right) \left( \frac{Y - \mu_Y}{\sigma_Y} \right) \right]$$

- (a)  $-1 \leq \rho_{XY} \leq 1$ ;
- (b)  $\rho_{XY}$  is either +1 or -1 if and only if X and Y are linear functions of one another.

If random variables X and Y are independent, then

$$E(g(X) \cdot h(Y)) = E(g(X)) \cdot E(h(Y)).$$

$$\Rightarrow \text{Cov}(X, Y) = \sigma_{XY} = 0, \quad \text{Corr}(X, Y) = \rho_{XY} = 0.$$

**Example 1:**

Consider the following joint probability distribution  $p(x, y)$  of two random variables X and Y:

	y			
x	0	1	2	$p_X(x)$
1	0.15	0.10	0	0.25
2	0.25	0.30	0.20	0.75
$p_Y(y)$	0.40	0.40	0.20	1.00

Recall:

$$E(X) = 1.75,$$

$$E(Y) = 0.8,$$

$$E(XY) = 1.5.$$

Find  $\text{Cov}(X, Y) = \sigma_{XY}$  and  $\text{Corr}(X, Y) = \rho_{XY}$ .

**Example 2:**

Let the joint probability density function for  $(X, Y)$  be

$$f(x, y) = \begin{cases} 60x^2y & 0 \leq x \leq 1, 0 \leq y \leq 1, x + y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find  $\text{Cov}(X, Y) = \sigma_{XY}$  and  $\text{Corr}(X, Y) = \rho_{XY}$ .

Recall:  $f_X(x) = 30x^2(1-x)^2, \quad 0 < x < 1, \quad E(X) = \frac{1}{2},$

$f_Y(y) = 20y(1-y)^3, \quad 0 < y < 1, \quad E(Y) = \frac{1}{3}, \quad E(XY) = \frac{1}{7}.$

**Example 3:**

Let the joint probability density function for  $(X, Y)$  be

$$f(x, y) = \begin{cases} x + y & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find  $\text{Cov}(X, Y) = \sigma_{XY}$  and  $\text{Corr}(X, Y) = \rho_{XY}$ .

Recall:  $f_X(x) = x + \frac{1}{2}, \quad 0 < x < 1. \quad f_Y(y) = y + \frac{1}{2}, \quad 0 < y < 1.$

**Example 4:**

Let the joint probability density function for  $(X, Y)$  be

$$f(x, y) = \begin{cases} 12x(1-x)e^{-2y} & 0 \leq x \leq 1, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Find  $\text{Cov}(X, Y) = \sigma_{XY}$  and  $\text{Corr}(X, Y) = \rho_{XY}$ .

Recall:  $f_X(x) = 6x(1-x), 0 < x < 1.$   $f_Y(y) = 2e^{-2y}, y > 0.$