STAT 400
UIUC

5.3 – Several Independent Random Variables

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If $X_1, X_2, \dots, X_n$ are n random variables and $a_0, a_1, a_2, \dots, a_n$ are $n+1$ constants, then the random variable $U = a_0 + a_1 X_1 + a_2 X_2 + \dots + a_n X_n$ has mean

$$E(U) = a_0 + a_1 E(X_1) + a_2 E(X_2) + \dots + a_n E(X_n)$$

and variance

$$\text{Var}(U) = \sum_{i=1}^{n} a_i^2 \text{Var}(X_i) + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ i \neq j}}^{n} a_i a_j \text{Cov}(X_i, X_j)$$

$$= \sum_{i=1}^{n} a_i^2 \text{Var}(X_i) + 2 \sum_{i<j} \sum a_i a_j \text{Cov}(X_i, X_j)$$

If $X_1, X_2, \dots, X_n$ are n independent random variables and $a_0, a_1, a_2, \dots, a_n$ are $n+1$ constants, then the random variable $U = a_0 + a_1 X_1 + a_2 X_2 + \dots + a_n X_n$ has variance

$$\text{Var}(U) = a_1^2 \text{Var}(X_1) + a_2^2 \text{Var}(X_2) + \dots + a_n^2 \text{Var}(X_n)$$

If $X_1, X_2, \dots, X_n$ are n independent random variables and $a_0, a_1, a_2, \dots, a_n$ are $n+1$ constants, and $U = a_0 + a_1 X_1 + a_2 X_2 + \dots + a_n X_n$, then

$$M_U(t) = e^{a_0 t} \cdot M_{X_1}(a_1 t) \cdot M_{X_2}(a_2 t) \cdot \dots \cdot M_{X_n}(a_n t).$$

If $X_1, X_2, \dots, X_n$ are normally distributed random variables,
then U is also normally distributed.

Example 1:
Models of the pricing of stock options often make the assumption of a normal distribution. An investor believes that the price of an *Burger Queen* stock option is a normally distributed random variable with mean \$18 and standard deviation \$3. He also believes that the price of an *Dairy King* stock option is a normally distributed random variable with mean \$14 and standard deviation \$2. Assume the stock options of these two companies are independent.
The investor buys 8 shares of *Burger Queen* stock option and 9 shares of *Dairy King* stock option. What is the probability that the value of this portfolio will exceed \$300?

Example 2:
A machine fastens plastic screw-on caps onto containers of motor oil. If the machine applies more torque than the cap can withstand, the cap will break. Both the torque applied and the strength of the caps vary. The capping machine torque has the normal distribution with mean 7.9 inch-pounds and standard deviation 0.9 inch-pounds. The cap strength (the torque that would break the cap) has the normal distribution with mean 10 inch-pounds and standard deviation 1.2 inch-pounds. The cap strength and the torque applied by the machine are independent.

What is the probability that a cap will break while being fastened by the capping machine?
That is, find the probability P (Strength < Torque).

Example 3:

In Neverland, the weights of adult men are normally distributed with mean of 170 pounds and standard deviation of 10 pounds, and the weights of adult women are normally distributed with mean of 125 pounds and standard deviation of 8 pounds. Six women and four men got on an elevator. Assume that all their weights are independent.

What is the probability that their total weight exceeds 1500 pounds?

<u>Example 4</u>:

Let X and Y be two independent Poisson random variables with mean λ_1 and λ_2 , respectively.

Let $W = X + Y$.

a) What is the probability distribution of W?

b) What is the conditional distribution of X given $W = n$?

If random variables X and Y are independent, then

$$M_{X+Y}(t) = M_X(t) \cdot M_Y(t).$$

If X and Y are independent,

X is Bernoulli(p), Y is Bernoulli(p) $\Rightarrow$ X + Y is Binomial$(n = 2, p)$;

X is Binomial(n_1, p), Y is Binomial(n_2, p) $\Rightarrow$ X + Y is Binomial$(n_1 + n_2, p)$;

X is Geometric(p), Y is Geometric(p) $\Rightarrow$ X + Y is Neg. Binomial$(r = 2, p)$;

X is Neg. Binomial(r_1, p), Y is Neg. Binomial(r_2, p)

$\Rightarrow$ X + Y is Neg. Binomial$(r_1 + r_2, p)$;

X is Poisson(λ_1), Y is Poisson(λ_2) $\Rightarrow$ X + Y is Poisson$(\lambda_1 + \lambda_2)$;

X is Exponential(θ), Y is Exponential(θ) $\Rightarrow$ X + Y is Gamma$(\alpha = 2, \theta)$;

X is Gamma(α_1, θ), Y is Gamma(α_2, θ) $\Rightarrow$ X + Y is Gamma$(\alpha_1 + \alpha_2, \theta)$;

X is Normal(μ_1, σ_1^2), Y is Normal(μ_2, σ_2^2)

$\Rightarrow$ X + Y is Normal$(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.