6.4 - Maximum Likelihood Estimation, Method of Moments

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p.m.f. or p.d.f. $f(x;\theta)$, $\theta \in \Omega$. Ω – parameter space.

Example 1: Suppose $\Omega = \{1, 2, 3\}$ and the p.d.f. $f(x;\theta)$ is

$\theta = 1$: $f(1;1) = 0.6$, $f(2;1) = 0.1$, $f(3;1) = 0.1$, $f(4;1) = 0.2$.

$\theta = 2$: $f(1;2) = 0.2$, $f(2;2) = 0.3$, $f(3;2) = 0.3$, $f(4;2) = 0.2$.

$\theta = 3$: $f(1;3) = 0.3$, $f(2;3) = 0.4$, $f(3;3) = 0.2$, $f(4;3) = 0.1$.

What is the maximum likelihood estimate of θ (based on only one observation of X) if …

a) $X = 1$;

b) $X = 2$;

c) $X = 3$;

d) $X = 4$.

Likelihood function:

$$L(\theta) = L(\theta; x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f(x_i; \theta) = f(x_1; \theta) \cdot \ldots \cdot f(x_n; \theta)$$

It is often easier to consider $\ln L(\theta) = \sum_{i=1}^{n} \ln f(x_i; \theta)$.

Maximum Likelihood Estimator: $\hat{\theta} = \arg\max L(\theta) = \arg\max \ln L(\theta)$.

Method of Moments:

$E(X) = g(\theta)$. Set $\overline{X} = g(\tilde{\theta})$. Solve for $\tilde{\theta}$.

<u>Example 0</u>:

Consider a single observation X of a Binomial random variable with n trials and probability of "success" p. That is,

$$P(X = k) = {}_nC_k \, p^k (1-p)^{n-k}, \qquad k = 0, 1, \ldots, n.$$

a) Obtain the method of moments estimator of p, $\tilde{p}$.

b) Obtain the maximum likelihood estimator of p, $\hat{p}$.

Example 2:

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a Poisson distribution with mean λ, $\lambda > 0$. That is,

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}, \quad k = 0, 1, 2, 3, \ldots .$$

a) Obtain the method of moments estimator of λ, $\tilde{\lambda}$.

b) Obtain the maximum likelihood estimator of λ, $\hat{\lambda}$.

Example 3:
Let $X_1, X_2, \dots, X_n$ be a random sample of size n from a Geometric distribution with probability of "success" p, $0 < p < 1$. That is,

$$P(X = k) = (1 - p)^{k-1} p, \quad k = 1, 2, 3, \dots .$$

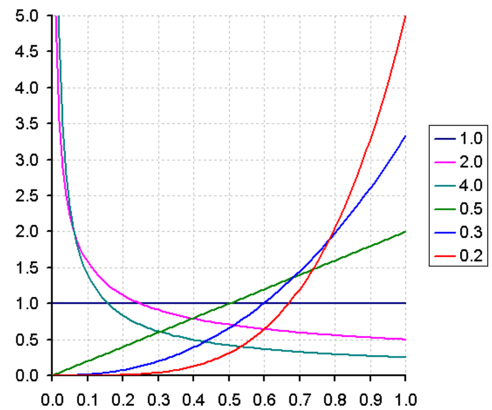
a) Obtain the method of moments estimator of p, $\tilde{p}$.

b) Obtain the maximum likelihood estimator of p, $\hat{p}$.

Example 4:
Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the distribution with probability density function

$$f(x;\theta) = \begin{cases} \dfrac{1}{\theta} \cdot x^{\frac{1-\theta}{\theta}} & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$0 < \theta < \infty$.



a) Obtain the method of moments estimator of θ, $\tilde{\theta}$.

Method of Moments:

$E(X) = g(\theta)$. Set $\overline{X} = g(\tilde{\theta})$. Solve for $\tilde{\theta}$.

b) Obtain the maximum likelihood estimator of θ, $\hat{\theta}$.

Likelihood function:

$$L(\theta) = L(\theta; x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f(x_i;\theta) = f(x_1;\theta) \cdot \ldots \cdot f(x_n;\theta)$$

Maximum Likelihood Estimator: $\hat{\theta}$ = arg max $L(\theta)$ = arg max ln $L(\theta)$.

c) Suppose $n = 3$, and $x_1 = 0.2$, $x_2 = 0.3$, $x_3 = 0.5$. Compute the values of the method of moments estimate and the maximum likelihood estimate for θ.

Example 5:

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from $N(\theta_1, \theta_2)$, where

$\Omega = \{ (\theta_1, \theta_2) : -\infty < \theta_1 < \infty,\ 0 < \theta_2 < \infty \}$. That is, here we let $\theta_1 = \mu$ and $\theta_2 = \sigma^2$.

a) Obtain the maximum likelihood estimator of θ_1, $\hat{\theta}_1$, and of θ_2, $\hat{\theta}_2$.

b) Obtain the method of moments estimator of θ_1, $\tilde{\theta}_1$, and of θ_2, $\tilde{\theta}_2$.